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ON THE THERMODYNAMIC INTERPRETATION OF THE EVOLUTIONARY CONDITIONS OF THE EQUATIONS OF THE MECHANICS OF FINITELY DEFORMABLE VISCOELASTIC MEDIA OF MAXWELL TYPE

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A thermodynamic interpretation is given of the phenomenon of the loss of evolutionarity in the hydrodynamics equations of viscoelastic incompressible fluids corresponding to models proposed in [1, 2].

A Clausius inequality is formulated for the virtual perturbations of the equilibrium parameters on the basis of the second law of thermodynamics and propositions on the local thermodynamic equilibrium in a small particle of a continuous medium [3].

Properties of reversible instantaneous deformations in the considered media are investigated and the form of the integral energy is found. The internal energy in the Oldroyd models [1] depends on the first invariant of the viscoelastic stress tensor, but can also be expressed in terms of the reversible strain components. In the De Witt model [2] it depends on the second invariant of the stress tensor and is nonlocal relative to the reversible strain.

Necessary conditions for the thermodynamic stability are obtained by using the expressions found for the internal energy and the Clausius inequality. Constraints on the principal values of the viscoelastic stress tensor result from these conditions which have been established earlier on the basis of demands for the evolutionarity of the corresponding systems of hydrodynamics equations [4, 5].

1. Clausius inequality as a requirement for the stability of the local thermodynamic equilibrium of an element of a continuous

medium. In describing the processes taking place at a fixed material particle, let us use the Lagrange coordinate system ξ^i with moving covariant basis $\mathbf{э}_i\ (t)$ and contravariant basis $\mathbf{э}^i\ (t)$. The metric tensors in the initial and strain states are defined by the formulas [3]

$$\mathbf{G}_0 = g_{ij}{}^{\circ}\mathbf{э}_0{}^i\mathbf{э}_0{}^j = g_0{}^{ij}\mathbf{э}_i{}^{\circ}\mathbf{э}_j{}^{\circ}, \quad \mathbf{G} = g_{ij}\mathbf{э}^i\mathbf{э}^j = g^{ij}\mathbf{э}_i\mathbf{э}_j, \quad \mathbf{э}_i{}^{\circ} = \mathbf{э}_i\,(t_0), \quad \mathbf{э}_0{}^i = \mathbf{э}^i\,(t_0)$$

$$g_{ij}{}^{\circ} = (\mathbf{э}_i{}^{\circ}, \mathbf{э}_j{}^{\circ}), \quad g_0{}^{ij} = (\mathbf{э}_0{}^i, \mathbf{э}_0{}^j), \quad g_{ij} = (\mathbf{э}_i, \mathbf{э}_j), \quad g^{ij} = (\mathbf{э}^i, \mathbf{э}^j)$$

Measuring the magnitude of the strain from the instant $t = t_0$, let us introduce the Green's finite strain tensors

$$\boldsymbol{\varepsilon}_* = \varepsilon_{ij*}\mathbf{э}^i\mathbf{э}^j = \varepsilon_*{}^{ij}\mathbf{э}_i\mathbf{э}_j, \qquad \boldsymbol{\varepsilon}^* = \varepsilon^{ij*}\mathbf{э}_i\mathbf{э}_j = \varepsilon_{ij}{}^*\mathbf{э}^i\mathbf{э}^j$$

$$\varepsilon_{ij*} = {}^1/_2(g_{ij} - g_{ij}{}^{\circ}), \qquad \varepsilon^{ij*} = {}^1/_2(g_0{}^{ij} - g^{ij}) \tag{1.1}$$

The tensors $\boldsymbol{\varepsilon}_*$ and $\boldsymbol{\varepsilon}^*$ are distinct. Besides these tensors, let us use the Hencky finite strain tensor

$$\mathbf{H} = {}^1/_2 \ln(\mathrm{G} + 2\boldsymbol{\varepsilon}^*) = -{}^1/_2 \ln(\mathrm{G} - 2\boldsymbol{\varepsilon}_*)$$

The principal values of the tensors $\boldsymbol{\varepsilon}_*$, $\boldsymbol{\varepsilon}^*$ and $\mathbf{H}$ denoted, respectively, by ε_{i*}, $\varepsilon_i{}^*$ and h_i, are interrelated by the formulas

$$\varepsilon_{i*} = {}^1/_2(1 - e^{-2h_i}), \qquad \varepsilon_i{}^* = {}^1/_2(e^{2h_i} - 1), \quad (1 + 2\varepsilon_i{}^*)(1 - 2\varepsilon_{i*}) = 1 \tag{1.2}$$

We represent the stress tensor $\mathbf{P}$ as

$$\mathbf{P}(\xi^i, t) = p^{ij}\mathbf{э}_i\mathbf{э}_j = p_{\cdot j}{}^{i\cdot}\mathbf{э}_i\mathbf{э}^j = p_{i\cdot}{}^{\cdot j}\mathbf{э}^i\mathbf{э}_j = p_{ij}\mathbf{э}^i\mathbf{э}^j \tag{1.3}$$

Let us examine some consequences of the laws of thermodynamics not related to a specific choice of the model of the continuous medium. Following [3], we assume that the state of a physically infinitesimal macroscopic volume at an arbitrary time can be considered as thermodynamic equilibrium. This state is characterized by a set of governing parameters on which the internal energy and entropy depend in a unique manner. We assume that the thermodynamic state of the medium is determined by assigning the stress tensor $\mathbf{P}$ and the absolute temperature T.

Let us examine the virtual perturbations of the equilibrium prameters which transfer a particle into the state $(\mathbf{P}_0 + \delta\mathbf{P},\ T_0 + \delta T)$ with the value $U_0 + \delta U$ for the specific internal energy and $s_0 + \delta s$ for the specific entropy. We shall limit ourselves to an analysis of the local perturbations by considering the thermodynamic parameters on the outer boundary of the particle to equal the unperturbed parameters $\mathbf{P}_0$, T_0. If the initial state of a small particle with the parameters $\mathbf{P}_0$, T_0 is stable, then in the case of sufficiently small perturbations this state should be reproduced because of the spontaneous process of interaction with the external medium.

Let us take the traditional scheme in classical thermodynamics by replacing the medium surrounding the particle by a thermostat with the invariant parameters $\mathbf{P}_0$, T_0 and considering that the thermostat together with the particles form an adiabatically insulated system [6, 7]. Therefore, the mechanical effect of the thermostat on the particle reduces to the effect of the external surface forces on its boundary, governed by the tensor $\mathbf{P}_0$. The process of building up complete thermodynamic equilibrium in the particle-thermostat system will be irreversible. The transition $(\mathbf{P}_0 + \delta\mathbf{P},\ T_0 + \delta T) \rightarrow (\mathbf{P}_0,\ T_0)$ hence occurs, and therefore the transition $(U_0 + \delta U,\ s_0 + \delta s) \rightarrow (U_0,\ s_0)$.

The classical Clausius inequality $\Delta s > \Delta Q\ /\ T_0$ is valid for an irreversible process, where Δs and ΔQ are the change in entropy and the quantity of heat absorbed from the thermostat per unit mass of the particle. Using the first law of thermodynamics, we reduce the Clausius inequality to

$$\Delta Q = \Delta U - \Delta A_e < T_0 \Delta s \qquad (1.4)$$

Here ΔA_e is the work per unit mass of the external surface forces on the strains induced by the transition $(\mathbf{P}_0 + \delta\mathbf{P},\ T_0 + \delta T) \rightarrow (\mathbf{P}_0,\ T_0)$. The real process of equilibrium build-up in the system proceeds in the direction $(U_0 + \delta U,\ s_0 + \delta s) \rightarrow (U_0,\ s_0)$, hence we replace the true by virtual increments in the last inequality, hence reversing the sign of the inequality

$$\delta U - \delta A_e - T_0 \delta s > 0 \qquad (1.5)$$

This inequality is basic to the investigation of the stability of local thermodynamic equilibrium.

In passing from (1.4) to (1.5) it was assumed that $\Delta A_e = -\delta A_e$. This is true in the case of an irreversible thermodynamic build-up process in which the strains during the transition $(\mathbf{P}_0 + \delta\mathbf{P},\ T_0 + \delta T) \rightarrow (\mathbf{P}_0,\ T_0)$ differ only in sign from the strains corresponding to the reverse transition. Such strains should be considered reversible, keeping in mind that they are indeed realized in the reversible process connecting the equilibrium states $(\mathbf{P}_0 + \delta\mathbf{P},\ T_0 + \delta T)$ and $(\mathbf{P}_0,\ T_0)$. If the work ΔA_e is not determined by the magnitude of the final strain and depends on the strain history, then the equality $\Delta A_e = -\delta A_e$ is satisfied when all the same strain stages occur in the reverse direction as in the direct process.

The existence and nature of the reversible strains depend on the specific model of the medium. For example, in a viscous incompressible fluid the reversible strains are generally missing, and we must set $\delta A_e = 0$ in (1.5). In an ideal gas, any strain reducing to a change in density can be achieved by reversible means, and in this case (1.5) becomes

$$\delta U + p_0 \delta (1 / \rho) - T_0 \delta s > 0 \qquad (1.6)$$

This inequality has been discussed repeatedly in the literature [6, 7]. In particular, the condition of the minimum internal energy in the stable equilibrium state with respect to processes with invariant density and entropy results therefrom. The possibility of using this extremal property to determine the stability conditions in a multicomponent gas mixture was indicated in [3]. These ideas have been developed in [8].

Thermodynamic inequalities assuring the stability of local equilibrium have usually been established for an ideal gas model. In particular, the equilibrium condition $(\partial p / \partial \rho)_s > 0$, which is simultaneously the evolutionary condition for the gasdynamics equations [9], follows from (1.6). This latter circumstance permits the assumption that the evolutionary conditions of the equations of mechanics of certain other media can also be obtained by starting from the demands for thermodynamic stability.

In deriving the stability conditions from (1.5) it is necessary to know the expression for the virtual work δA_e and for the variation in the internal energy δU.

It can be shown that when the external surface forces on the particle boundary are determined by the invariant tensor $\mathbf{P}_0$ during strain, the work element in a time dt has the form

$$dA_e = (2\rho)^{-1} p_*^{ij}\, dg_{ij} \qquad (1.7)$$

In this formula p_*^{ij} are the components of the constant tensor $\mathbf{P}_0$ in the variable basis $\mathbf{э}_i$. The increments dg_{ij} should only be associated with the reversible strains. In the simplest case, when triaxial compression-tension occurs along the principal axes of the tensor $\mathbf{P}_0$ in all stages of the strain process, (1.7) reduces to

$$dA_e = \frac{1}{3\rho}\operatorname{Sp}\mathbf{P}_0\,dH_1 + \frac{1}{\rho}\sum_{i=1}^{3} p_i{}^{\circ}\,dh_i{}'$$

$$H_1 = \operatorname{Sp}\mathbf{H},\qquad \mathbf{H}' = \mathbf{H} - {}^1/_3 H_1\mathbf{G},\qquad \rho = \rho_0 e^{-H_1} \tag{1.8}$$

Here $\mathbf{H}$ is the Hencky reversible strain tensor, $\mathbf{H}'$ is the deviator of the Hencky tensor, $p_i{}^{\circ}$ and $h_i{}'$ are the principal values of the tensor $\mathbf{P}_0$ and $\mathbf{H}'$ respectively.

In the general case the expression (1.8) for dA_e is not a total differential, and the total virtual work

$$\delta A_e = \int_{g_{ij}{}^{\circ}}^{g_{ij}{}^{\circ}+\delta g_{ij}} dA_e$$

depends on the strain process. However, in two important particular cases δA_e is determined by the final strains.

The first case corresponds to the global tensor $\mathbf{P}_0 = -p_0\mathbf{G}_0$. Then the sum in the right side of (1.8) equals zero, and integrating we arrive at the simple formula: $\delta A_e = -p_0\delta\,(1/\rho)$, which always holds for an ideal fluid.

The second case corresponds to the incompressible medium: $\rho = \rho_0$, $\mathbf{H}' = \mathbf{H}$. Then

$$\delta A_e = \frac{1}{\rho}\sum_{i=1}^{3} p_i{}^{\circ}\delta h_i$$

Precisely this case is of interest for the viscoelastic fluid models to be examined below. Taking account of the last formula for δA_e, the inequality (1.5) applied to a coaxial strain process can be written as

$$\delta U - \frac{1}{\rho}\sum_{i=1}^{3} \pi_i{}^{\circ}\delta h_i - T_0\delta s > 0 \tag{1.9}$$

Here the quantities $\pi_i{}^0$ equal to the principal values of the tensor $\mathbf{\Pi}_0 = \mathbf{P}_0 + p_0\mathbf{G}_0$ are used in place of $p_i{}^0$. Such a substitution is admissible because of the condition $\delta H_1 = 0$.

2. Instantaneous strains and internal energy in viscoelastic Maxwellian media. The stress tensor in the viscoelastic fluids under consideration is

$$\mathbf{P} = -p\mathbf{G} + \mathbf{\Pi},\qquad \mathbf{\Pi} = \pi^{ij}\text{э}_i\text{э}_j = \pi_{\cdot j}{}^{i\cdot}\text{э}_i\text{э}^j = \pi_{i\cdot}{}^{\cdot j}\text{э}^i\text{э}_j = \pi_{ij}\text{э}^i\text{э}^j$$

In the general case the viscoelastic stress tensor $\mathbf{\Pi}$ is not a deviator. We denote the principal values of the tensor $\mathbf{\Pi}$ by π_i in decreasing order so that $\pi_1 \geqslant \pi_2 \geqslant \pi_3$.

In Lagrange coordinates, the rheological equations of the Oldroyd "contravariant" and "covariant" models are the following, respectively:

$$\frac{d\pi^{ij}}{dt} + \frac{1}{\lambda}\pi^{ij} = -\mu\frac{dg^{ij}}{dt},\qquad \frac{d\pi_{ij}}{dt} + \frac{1}{\lambda}\pi_{ij} = \mu\frac{dg_{ij}}{dt}\qquad \left(\mu = \frac{\eta}{\lambda}\right) \tag{2.1}$$

The De Witt model is described by the rheological equation

$$\frac{1}{2}\left(\frac{d\pi_{ij}}{dt} + g_{ir}g_{js}\frac{d\pi^{rs}}{dt}\right) + \frac{1}{\lambda}\pi_{ij} = \mu\frac{dg_{ij}}{dt} \tag{2.2}$$

The symbol d/dt in these equations denotes differentiation with respect to time for fixed Lagrange coordinates. Integrating (2.1), (2.2) with the initial condition $\Pi(\xi^i, t_0) = \Pi_0(\xi^i)$, we can obtain hereditary relationships governing the state of stress at the time t in terms of the initial tensor Π_0 and the strain history $g_{ij}(t')$ in the interval $t_0 < t < t$.

The instantaneous strains corresponding to a jump change in the basis vectors $\mathfrak{s}_i$ can be considered in viscoelastic liquids of Maxwell type. This property of media possessing instantaneous elasticity predicts the linear theory of viscoelasticity [10] in particular. The concept of instantaneous strain in viscoelastic media in the presence of finite strain is manifested in [11, 12].

Let us establish the general form of the relation between the stresses and the instantaneous strains for the Oldroyd model. Integrating the rheological equations (2.1) we obtain

$$\pi^{ij}(t) = \pi^{ij}(t_0)\exp\left(\frac{t_0 - t}{\lambda}\right) - \mu\int_{t_0}^{t}\exp\left(\frac{t' - t}{\lambda}\right)\frac{dg^{ij}(t')}{dt'}dt' \tag{2.3}$$

$$\pi_{ij}(t) = \pi_{ij}(t_0)\exp\left(\frac{t_0 - t}{\lambda}\right) + \mu\int_{t_0}^{t}\exp\left(\frac{t' - t}{\lambda}\right)\frac{dg_{ij}(t')}{dt'}dt'$$

We make the formal substitution $t_0 \to t - \Delta t$, $t \to t + \Delta t$ in (2.3) and examine a continuous law of variation of the metric tensor components $g_{ij} = g_{ij}(t', \Delta t)$ such that the limit functions

$$g_{ij}(t') = \lim_{\Delta t \to 0} g_{ij}(t', \Delta t)$$

would be discontinuous at $t' = t$ and hence $\{g_{ij}\} = g_{ij}(t + 0) - g_{ij}(t - 0)$. Then passing to the limit as $\Delta t \to 0$ in (2.3), we obtain for the contravariant and covariant models, respectively,

$$\begin{gathered}\pi^{ij} = \pi_0^{ij} + 2\mu\varepsilon^{ij*}\\ \pi^{ij} = \pi^{ij}(t + 0), \quad \pi_0^{ij} = \pi^{ij}(t - 0), \quad 2\varepsilon^{ij*} = -\{g^{ij}\}\\ \pi_{ij} = \pi_{ij}^{\circ} + 2\mu\varepsilon_{ij*}\\ \pi_{ij} = \pi_{ij}(t + 0), \quad \pi_{ij}^{\circ} = \pi_{ij}(t - 0), \quad 2\varepsilon_{ij*} = \{g_{ij}\}\end{gathered} \tag{2.4}$$

Let us note that the relationships between the stresses and instantaneous strains (2.4) are exactly the relationships between the stresses and strains in hypoelastic media. The equations of hypoelastic media are obtained by discarding the terms π^{ij} / λ or π_{ij} / λ in (2.1). The discarded terms model the viscous components in the Maxwell elements and are negligible for instantaneous strains since a finite time is necessary for the development of viscous strain.

Let the initial Lagrange basis be orthonormal and directed along the principal axes of the tensor Π_0. Then it is seen from (2.4) that the matrices $\|\pi^{ij}\|$ or $\|\pi_{ij}\|$ are diagonal if and only if the initial orthonormal basis $\mathfrak{s}_i^{\circ}$ directed along the principal axes of the tensor Π_0 goes over into the orthogonal basis $\mathfrak{s}_i$ as a result of strain. Therefore, the principal stress and strain axes may be distinct after the instantaneous strain. This fact dropped out of the field of view of investigations on the instantaneous strains because of the tendency to consider an isotropic initial state of stress [11]. Meanwhile, an anisotropic state of stress of a viscoelastic material influences its subsequent behavior strongly [5], especially for instantaneous strains when the initial state is not successfully obliterated in the memory of the medium.

In the case of the isotropic tensor $\Pi_0 = \pi_0 G_0$ the principal axes of the tensors Π,

$\boldsymbol{\varepsilon}^*$ and $\boldsymbol{\varepsilon}_*$ coincide. Only the diagonal elements of the matrices $\| \pi^{ij} \|$ and $\| \varepsilon^{ij*} \|$ as well as the matrices $\| \pi_{ij} \|$ and $\| \varepsilon_{ij*} \|$ can differ from zero in the basis $\mathbf{э}_i$ directed along these principal axes. In this case we obtain, respectively, from (2.4)

$$\pi_i = \pi_0 + 2(\mu + \pi_0)\varepsilon_i{}^* = \pi_0 + (\mu + \pi_0)(e^{2h_i} - 1)$$

$$\pi_i = \pi_0 + 2(\mu - \pi_0)\varepsilon_{i_*} = \pi_0 + (\mu - \pi_0)(1 - e^{-2h_i})$$

In deducing the relationships written down we have used the rules of passing from the contravariant and covariant components of the tensor $\mathbf{\Pi}$ in the orthogonal bases $\mathbf{э}_i$ and $\mathbf{э}^i$ to the principal values of this tensor

$$\pi_\alpha = \pi^{\alpha\alpha} g_{\alpha\alpha} = \pi^{\alpha\alpha}(1 + 2\varepsilon_\alpha{}^*), \qquad \pi_\alpha = \pi_{\alpha\alpha} g^{\alpha\alpha} = \pi_{\alpha\alpha}(1 - 2\varepsilon_{\alpha*})$$

Summation is not carried out over the repeated indices in these latter formulas.

It is seen from the relationships established that in the case $\pi_0 < -\mu$ the additional compressive stress resultant in the principal direction $\pi_i < \pi_0$ should result in elastic elongation of the fiber in the material corresponding to the contravariant model. In the case $\pi_0 > \mu$ the additional tensile stress resultant $\pi_i > \pi_0$ will result in contraction of the principal fiber in the covariant model. The result obtained does not agree with the customary properties of elastic media although it also does not contradict the laws of mechanics. It will be seen later that such an anomalous behavior of viscoelastic media is forbidden by the stability requirements for local thermodynamic equilibrium.

Let us assume that the instantaneous strains do not violate the initially Euclidean metric of the space connected with the material continuum. Then the principal values of the instantaneous strain tensors $\boldsymbol{\varepsilon}^*$, $\boldsymbol{\varepsilon}_*$ and $\mathbf{H}$ should lie within the following limits [3]:

$$\varepsilon_i{}^* > -{}^1/_2, \qquad \varepsilon_{i_*} < {}^1/_2, \qquad -\infty < h_i < \infty$$

These inequalities for the principal values of the instantaneous strain tensors are the limit results of analogous inequalities which must be satisfied for arbitrarily fast continuous finite strains. Measuring the components of the tensor $\mathbf{H}$ from the state corresponding to an isotropic viscoelastic pressure π_0, we obtain

$$e^{\pm 2h_i} = (\pi_i \pm \mu)/(\pi_0 \pm \mu) > 0 \tag{2.5}$$

Here and henceforth, the upper signs correspond to the contravariant, and the lower to the covariant models.

Using the condition of incompressibility of the instantaneous strains, we can uniquely determine the isotropic viscoelastic pressure π_0 which can be achieved by instantaneous strain from a state with given tensor $\mathbf{\Pi}$

$$\pi_0 = \mp \mu + [(\pi_1 \pm \mu)(\pi_2 \pm \mu)\ (\pi_3 \pm \mu)]^{1/3} \tag{2.6}$$

Some of the thermodynamic potentials are usually given as a function of the corresponding set of equilibrium parameters [3] in a thermodynamic derivation of the equations of state of continuous media. The problem of seeking the inner or free energy by means of given equations of state arises in postulating the governing equations. This problem reduces to integrating the Gibbs equation

$$dU = -dA_i{}^r + T ds = \rho^{-1}\pi^{ij}\, d\varepsilon_{ij*} + T ds$$

Here $d\varepsilon_{ij*}$ are the increments of the covariant components of the reversible strain tensor $\boldsymbol{\varepsilon}_*$ and $dA_i{}^r$ is the work element of the internal stresses on the reversible strains.

The rheological equations (2.1), (2.2) do not contain the absolute temperature or entropy. In fact, these equations are used in that temperature range in which the rheological constants λ and η can be considred constant. The reversible work of the internal forces dA_i^r is independent of the entropy in this case, hence the quantity Tds must be a total differential, i.e. the entropy should depend only on the temperature and can be determined if the specific heat of an incompressible medium $c_V(T)$ is given.

The models (2.1), (2.2) are often used to describe the flows of solutions and polymer melts. Mention of the entropy nature of elasticity in some polymer fluids can be encountered in the literature. There is usually kept in mind here that in isothermal flows the change in internal energy is negligibly small and the reversible work of the internal stresses only result in a change in entropy. Then the relationships

$$dU = 0, \qquad -dF_{T_0} = T_0\, ds = dA_i^r \qquad (F = U - Ts)$$

are satisfied. Therefore, the change in entropy for constant internal energy ceases to be independent of the strain processes.

Let us emphasize that the nontrivial stability conditions of the thermodynamic state are determined essentially by the behavior of the reversible work of the internal stresses in the media under investigation. In fact, in the case of adiabatic virtual perturbations the Clausius inequality (1.5) becomes

$$\delta U_s - \delta A_e = -\delta A_i^r - \delta A_e > 0$$

If,however,the elasticity is of entropy nature, the inequality (1.5) can be written as

$$-\delta A_e - T_0 \delta s = -\delta A_e - \delta A_i^r > 0$$

In both cases precisely the analysis of the variations δA_i^r permits extraction of the class of unstable states.

It follows from the form of the rheological equations (2.1), (2.2) and the Gibbs relationship that the strain part, independent of the entropy, and the part dependent only on the entropy, enter additively into the expression for the internal energy.

Let us examine a reversible adiabatic instantaneous strain process for Oldroyd media. The reversible character of the instantaneous strains follows from (2.4). These relationships predict conservation of the length dl of an arbitrary material fiber in the jump-like "loading-unloading" process. The Gibbs equation for an infinitesimal adiabatic instantaneous strain is

$$dU_s = dF_T = (2\rho)^{-1} \pi^{ij}\, dg_{ij} \tag{2.7}$$

We obtain, respectively, for the contravariant and covariant models

$$\pi^{ij}\, dg_{ij} = d(\pi^{ij} g_{ij}) = d\Pi_1 \qquad (\Pi_1 = \operatorname{Sp} \Pi)$$
$$\pi^{ij}\, dg_{ij} = -\pi_{ij} dg^{ij} = -d(\pi_{ij} g^{ij}) = -d\Pi_1$$

The incompressibility condition $g^{ij} dg_{ij} = 0$ as well as the equality $g_{ij} d\pi^{ij} = 0$ or $g^{ij}\, d\pi_{ij} = 0$ following from the first or second group of the relationships (2.4) were used in deducing the last two formulas. Hence, (2.7) becomes

$$dU_s = dF_T = \pm (2\rho)^{-1} d\Pi_1$$

Therefore, the following formulas are valid for the finite changes in the strain parts of the internal and free energies in Oldroyd media:

$$\Delta U_s = \Delta F_T = \pm (2\rho)^{-1} \Delta \Pi_1 \tag{2.8}$$

Using (2.4), the ΔU_s can be expressed in terms of invariants of the initial tensor Π_0

and the strain tensors. For the contravariant and covariant models we obtain, respectively

$$\Delta U_s = \rho^{-1}\,\mathrm{Sp}\,(\mu\boldsymbol{\varepsilon}^* + \boldsymbol{\Pi}_0\boldsymbol{\varepsilon}_0), \qquad \boldsymbol{\varepsilon}_0 = \varepsilon^{ij*}\,\text{э}_0{}^i\text{э}_0{}^j$$
$$\Delta U_s = \rho^{-1}\,\mathrm{Sp}\,(\mu\boldsymbol{\varepsilon}_* - \boldsymbol{\Pi}_0\boldsymbol{\varepsilon}^\circ), \qquad \boldsymbol{\varepsilon}^\circ = \varepsilon^{ij*}\,\text{э}_i{}^\circ\text{э}_j{}^\circ$$

The principal values of the tensors $\boldsymbol{\varepsilon}_0$ and $\boldsymbol{\varepsilon}^\circ$ are $\varepsilon_i{}^*$ and ε_{i*}. Let $\text{э}_i{}^*$, $\text{э}_i{}^\circ$ be the orthonormal bases of the principal axes of the tensors $\boldsymbol{\Pi}_0$, $\boldsymbol{\varepsilon}_0$, respectively. The transformation from the basis $\text{э}_i{}^\circ$ to the basis $\text{э}_i{}^*$ be given by the orthogonal matrix $C = \| c_{\cdot j}{}^{i'} \|$, so that $\text{э}_i{}^* = c_{\cdot i}{}^{j'}\,\text{э}_j{}^\circ$. Then the expression for ΔU_s can be represented in the contravariant model as

$$\Delta U_s = \rho^{-1}\left(\mu \sum_{i=1}^{3} \varepsilon_i{}^* + \sum_{i,\,j=1}^{3} (c^{i\cdot}_{\cdot j})^2 \pi_j{}^\circ \varepsilon_i{}^*\right)$$

Therefore, ΔU_s depends not only on the principal values of the reversible strain tensor $\boldsymbol{\varepsilon}_0$, but also on the principal stresses $\pi_i{}^\circ$ in the initial state, as well as on the angles made by the principal axes of the tensors $\boldsymbol{\varepsilon}_0$ and $\boldsymbol{\Pi}_0$. (The nine components of the orthogonal matrix C depend on the three Euler angles governing the rotation of the trihedron $\text{э}_i{}^*$ relative to the trihedron $\text{э}_i{}^\circ$.) The ΔU_s for the covariant model can also be expressed in terms of the principal values of the tensors $\boldsymbol{\Pi}_0$ and $\boldsymbol{\varepsilon}^\circ$ and the angles governing the rotation of the principal strain axes relative to the initial principal stress axes. Measuring the reversible strain tensor $\mathbf{H}$ and the function U_s from the initial state with the isotropic tensor $\boldsymbol{\Pi}_0 = \pi_0 \mathbf{G}_0$, we obtain

$$U_s = \frac{1}{2\rho}(\mu \pm \pi_0)\left(e^{\pm 2h_1} + e^{\pm 2h_2} + e^{\pm 2h_3} - 3\right) \tag{2.9}$$

The connection between the stresses and the reversible strains can be established in a De Witt medium by Jaumann integration of the relationships (2.2) [13, 14] and subsequent passage to the limit to the instantaneous strain. Since the terms π_{ij} / λ in such a passage do not yield any contribution to the desired connection, the relationships between the stresses and instantaneous strains can be obtained by integrating the equations of a hypoelastic body

$$\frac{1}{2}\left(\frac{d\pi_{ij}}{dt} + g_{ir}g_{js}\frac{d\pi^{rs}}{dt}\right) = \mu\frac{dg_{ij}}{dt}$$

with a subsequent passage to the limit to the instantaneous strains. For infinitesimal increments in the matrices $\Pi = \| \pi_{\cdot j}{}^{i'} \|$ and $G = \| g_{ij} \|$ the equations of a hypoelastic medium are equivalent to the relationship

$$2\,d\Pi + G^{-1}dG\,\Pi - \Pi G^{-1}dG = 2\,\mu G^{-1}dG \tag{2.10}$$

The different matrix relation (2.10) is not holonomic. This is the essential distinction between the De Witt and the Oldroyd models, where the analogs of (2.10) are

$$d\,(\Pi\,G^{-1}) = -\,\mu dG^{-1}, \quad d\,(G\,\Pi) = \mu dG$$

and are total differential equations. Hence, the stresses in the model (2.2) are not defined uniquely by assigning the initial state of stress and the final reversible strain, but are defined uniquely by the history of the reversible strain. The strain history in the case of instantaneous strain should be understood to be the nature of the passage to the limit from the continuous to the instantaneous strain. In this sense there exists a dependence of the stress on the structure of the instantaneous strain.

We clarify the mentioned property of the De Witt model by examples which are useful for the subsequent exposition. Let the principal strain axes be unchanged in the material for all stages of the continuous strain. Let $(\pi_{ij}{}^\circ)$ denote the physical components of the tensor Π_0 in the basis $\text{э}_i{}^\circ$ and (π_{ij}) the physical components of the tensor Π in the basis э_i. Here $\text{э}_i{}^\circ$ and э_i are the initial and present bases of the principal strain axes. In the case under consideration, integration of (2.10) reduces to the formulas

$$(\pi_{\alpha\alpha}) = (\overset{\circ}{\pi}_{\alpha\alpha}) + 2\mu h_\alpha, \qquad (\pi_{\alpha\beta}) = (\overset{\circ}{\pi}_{\alpha\beta}) \qquad (\alpha \neq \beta) \tag{2.11}$$

These latter relationships do not depend explicitly on time and remain valid upon passage to the instantaneous strain. Formulas (2.11) are not true when the requirement of unchangeability of the principal strain axes in the material is violated.

Let us consider the continuous strain of simple shear which is characterized by the rotation of the principal strain axes in the material. For simplicity, let us limit ourselves to the case of plane stress and deformable state. We direct the orthonormal two-dimensional basis $\text{э}_i{}^\circ$ along the principal axes of the tensor Π_0 so that $\pi_1{}^\circ \geqslant \pi_2{}^\circ$. The vector $\text{э}_1(t)$ remains equal to э_1 during shear strain but the vector $\text{э}_2(t)$ is elongated and makes an angle $\varphi(t)$ with the vector $\text{э}_2{}^\circ$ so that the magnitude of the shear $a(t) = \operatorname{tg}\varphi(t)$. The state of stress at time t in a hypoelastic body is determined by integrating the matrix equation (2.10) with the given initial tensor Π_0 and the given strain history:

$$\Pi_0 = \begin{Vmatrix} \pi_1{}^\circ & 0 \\ 0 & \pi_2{}^\circ \end{Vmatrix}, \qquad G = \begin{Vmatrix} 1 & a \\ a & 1 + a^2 \end{Vmatrix} \qquad (a(t_0) = 0)$$

The matrix G satisfies the incompressibility condition: $\det G = 1$. Since G depends only on the magnitude of the shear $a(t)$, then taking $a(t)$ as a new independent variable we integrate (2.10), which is equivalent to a system of ordinary differential equations in the elements of the matrix Π. We hence obtain

$$\begin{aligned}
&\pi^{1\cdot}_{\cdot 1} = {}^1/_2\,(\pi_1{}^\circ + \pi_2{}^\circ) + \mu - [{}^1/_2\,(\pi_2{}^\circ - \pi_1{}^\circ) + \mu]\,(\cos a + a \sin a) \\
&\pi^{2\cdot}_{\cdot 1} = [{}^1/_2\,(\pi_2{}^\circ - \pi_1{}^\circ) + \mu] \sin a, \qquad \pi^{1\cdot}_{\cdot 2} = \pi^{2\cdot}_{1}\,(1 + a^2) + (\pi^{1\cdot}_{\cdot 1} - \pi^{2\cdot}_{\cdot 2})\, a \\
&\pi^{2\cdot}_{\cdot 2} = \pi_1{}^\circ + \pi_2{}^\circ - \pi^{1\cdot}_{\cdot 1}
\end{aligned} \tag{2.12}$$

These formulas are valid for the instantaneous strain if the matrix G has the form mentioned above during passage from the continuous to the instantaneous strain. However, the very same instantaneous shear strain can be obtained by passing to the limit from the continuous tension-compression strain along the principal axes of the tensor' ε_0 and subsequent quasisolid rotation. Then formulas (2.11) are valid, which yield a description of the final state of stress distinct from (2.12) under identical initial conditions for the stress and the very same magnitude of the final strain.

The universal formula for ΔU_s in the model (2.2) is obtained as follows. Keeping in mind the increments dg_{ij} in the reversible process, let us write (2.7) in the form

$$\rho dU_s = {}^1/_2\, \pi^{ij}\, dg_{ij} = {}^1/_2 \operatorname{Sp}(\Pi G^{-1} dG)$$

We multiply (2.10) by the matrix Π on the left, and we take the traces of both parts of the equation obtained by using the identity

$$\operatorname{Sp}(\Pi G^{-1} dG\, \Pi) = \operatorname{Sp}(\Pi^2 G^{-1} dG)$$

We hence obtain

$$\operatorname{Sp}(\Pi G^{-1} dG) = \frac{1}{\mu} \operatorname{Sp}(\Pi d\Pi) = \frac{1}{2\mu}\, d\Pi_2, \qquad \Pi_2 = \pi^{i}_{\cdot j} \pi^{j}_{\cdot i}$$

Consequently

$$\Delta U_s = \frac{1}{4\rho\mu}\Delta \Pi_2 \tag{2.13}$$

This formula is valid for any process in a De Witt medium where, as in the Oldroyd media, the thermodynamic state is defined by the viscoelastic stress tensor and the absolute temperature. Comparing (2.13) and (2.8), we see that in contrast to the Oldroyd model, the internal energy in the model with the Jaumann derivative depends on the second rather than the first invariant of the tensor Π. A greater distinction, in principle, is the nonlocal dependence of the internal energy on the reversible strains, i.e., the absence of a unique dependence of the internal energy in the medium (2.2) on the final reversible strain. This ambiguity can be eliminated only by involving data on the structure of the instantaneous strain.

In the particular case when passage to the limit from continuous strain with principal axes "frozen" in the material corresponds to the instantaneous strain, there follows from (2.11), (2.13)

$$\Delta U_s = \Delta F_T = \frac{\mu}{\rho}\,\Delta\,(h_1^2 + h_2^2 + h_3^2) \tag{2.14}$$

This formula has been presented in [14, 15]. In contrast to (2.13), it is not universal and is valid only for a particular form of the reversible strain process. In substance, the propositions on the coaxial relation between the tensor Π and the reversible strain tensor H and on the independence of the rheological behavior of the material from "elastic" rotations advanced in [14, 15] turn out to be more rigid than the consequences of the governing equations (2.2).

3. Thermodynamic derivation of the evolutionarity condition. We establish the necessary stability conditions for the thermodynamic state in the Oldroyd models. Let an arbitrary state with parameters Π_0, T_0 be given. We consider a perturbed state with parameters $\Pi_0 + \delta\Pi$, $T_0 + \delta T$ for which the reversible strain corresponding to the transition $\Pi_0 \to \Pi_0 + \delta\Pi$ reduces to tension-compression along the principal axes of the tensor Π_0. The Clausius inequality in the form (1.9) can be used in the case of coaxial deformation. The quantity δA_e is a linear form in the variation δh_i. At the same time, according to (2.8) the increment δU is a linear form in $\delta\pi_i$. However, there is a nonlinear relation between the virtual increments $\delta\pi_i$ and δh_i which follows from the relationships between the stresses and instantaneous strains. Hence, the expansion of the left side of (1.9) in either the independent variations δh_i or the independent variations $\delta\pi_i$ contains members of second and higher orders in these variations.

In the case under consideration it is convenient to expand the increment δU in the variable δh_i by using the representation (2.9) for the strain part of the internal energy. From the condition of incompressibility of the instantaneous strains there follows $h_3 = -(h_1 + h_2)$. Hence, for definiteness, let us consider $U = U(h_1, h_2, s)$. We expand the increment δU near the value $h_i = h_i^\circ$, $s = s_0$ by retaining second order terms in the expansion

$$\delta U = \sum_{i=1}^{2}\left(\frac{\partial U}{\partial h_i}\right)_0 \delta h_i + \left(\frac{\partial U}{\partial s}\right)_0 \delta s + \frac{1}{2}\sum_{i,\,j=1}^{2}\left(\frac{\partial^2 U}{\partial h_i \partial h_j}\right)_0 \delta h_i \delta h_j +$$
$$+ \sum_{i=1}^{2}\left(\frac{\partial^2 U}{\partial h_i \partial s}\right)_0 \delta h_i \delta s + \frac{1}{2}\left(\frac{\partial^2 U}{\partial s^2}\right)_0 (\delta s)^2$$

The zero subscript on the derivatives denotes that they are evaluated for the values $h_i = h_i^\circ$, $s = s_0$ corresponding to the unperturbed state. Using the expression (2.9), the relations between the quantities π_i°, h_i° and the isotropic unloading pressure π_0, as well as the condition $h_3 = -(h_1 + h_2)$ we obtain a formula valid for both Oldroyd models

$$\sum_{i=1}^{2} (\partial U/\partial h_i)_0 \delta h_i = \rho^{-1} [(\pi_1^\circ - \pi_3^\circ) \delta h_1 + (\pi_2^\circ - \pi_3^\circ) \delta h_2] \tag{3.1}$$

It is easy to verify that the right side of (3.1) equals δA_e. Moreover, using the condition $(\partial U/\partial s)_0 = T_0$, we see that the sum of the first order terms in (1.9) equals zero. Since $\partial^2 U / \partial h_i \partial s \equiv 0$, we arrive at the requirement

$$\sum_{i,\, j=1}^{2} \left(\frac{\partial^2 U}{\partial h_i \partial h_j}\right)_0 \delta h_i \delta h_j + \left(\frac{\partial^2 U}{\partial s^2}\right)_0 (\delta s)^2 > 0 \tag{3.2}$$

The conditions of positive definiteness of the quadratic form (3.2) are in the case of the contravariant and covariant models

$$\mu \pm \pi_0 > 0, \quad (dT / ds)_0 > 0 \tag{3.3}$$

Now taking account of (2.5), we see that compliance with the condition

$$\pi_3^\circ > -\mu \tag{3.4}$$

for the contravariant model, and

$$\pi_1^\circ < \mu \tag{3.5}$$

for the covariant model is necessary and sufficient for satisfaction of the first inequality in (3.3). The inequalities (3.4), (3.5) agree with the evolutionarity conditions of the hydrodynamics equations of corresponding Oldroyd media, which were established in [5] on the basis of an analysis of the dispersion equations for small perturbations.

Therefore, the possible nonevolutionarity of the dynamics equations of the Oldroyd media (2.1) is associated with the loss of stability of the local thermodynamic equilibrium. The condition $dT / ds > 0$ denotes positivity of the specific heat $c_V = Tds / dT$ of an incompressible material and is ordinarily satisfied for real media. The evolutionarity condition for the dynamics equations of the model (2.2) is also deduced from the Clausius inequality under somewhat different assumptions.

If the structure of the instantaneous strain is such that the principal axes of a rapidly changing continuous strain are always frozen in the material, Eq. (2.14) is valid for ΔU. The application of this formula to the Clausius inequality results in a trivial condition: $\mu > 0$. This result could be forseen on the basis of (2.11), which shows that anomalous elastic elongation of a fiber is impossible for frozen principal strain axes when the compressive stress resultant is increased.

Nonevolutionarity of the dynamics equations of a model with a Jaumann derivative can only appear in investigations of flows with rotation of the principal strain axes relative to the fluid. Hence, the reasons for nonevolutionarity of the equations corresponding to this model should be sought in the behavior of the internal energy under reversible strains, whose structure is associated with the rotation of the principal strain axes in the material.

Taking account of this latter remark, let us apply the inequality (1.5) to the virtual instantaneous strain obtained by passing to the limit from the continuous simple shear strain. For such a strain the change δU_s can be expressed in terms of the principal values of the tensor Π_0 and the magnitude of the shear a by using (2.12), (2.13). We hence obtain

$$\delta U_s = \rho^{-1} [{}^1/_2 (\pi_2{}^\circ - \pi_1{}^\circ) + \mu] (1 - \cos a) \qquad (3.6)$$

The virtual work δA_e on the reversible strains equals

$$\delta A_e = \frac{1}{2\rho} \int_{g_{ij}(0)}^{g_{ij}(a)} (\pi_1{}^\circ b^{i\cdot}_{\cdot 1} b^{j\cdot}_{\cdot 1} + \pi_2{}^\circ b^{i\cdot}_{\cdot 2} b^{j\cdot}_{\cdot 2})\, dg_{ij}$$

Here the matrix $\| b_{.j}{}^{i\cdot} \|$ determines the transition from the basis э_i to the basis $\text{э}_i{}^0$

$$\text{э}_i{}^\circ = b^{j\cdot}_{\cdot i} \text{э}_j, \qquad \| b^{i\cdot}_{\cdot j} \| = \begin{Vmatrix} 1 & -a \\ 0 & 1 \end{Vmatrix}$$

It is easy to verify that $\delta A_e = 0$ in the case under consideration. Keeping first and second order terms in the perturbations δa, δs in the δU expansion, we reduce the inequality (1.5) to

$$(\partial^2 U / \partial a^2)_0 (\delta a)^2 + (\partial^2 U / \partial s^2)_0 (\delta s)^2 > 0 \qquad (3.7)$$

The derivatives with zero subscript in (3.7) are evaluated for unperturbed values of the parameters $a = 0$, $s = s_0$. The conditions

$$\delta A_e = 0, \quad (\partial U / \partial a)_0 = 0, \quad (\partial U / \partial s)_0 = T_0, \quad \partial^2 U / \partial a \partial s \equiv 0$$

were used in deriving (3.7). Taking account of (3.6), we arrive at the requirement

$$\rho^{-1} [\mu - {}^1/_2 (\pi_1{}^\circ - \pi_2{}^\circ)] (\delta a)^2 + (dT / ds)_0 (\delta s^2) > 0$$

We hence obtain

$${}^1/_2 (\pi_1{}^\circ - \pi_2{}^\circ) < \mu, \qquad (dT / ds)_0 > 0 \qquad (3.8)$$

The first of the inequalities (3.8) limits the magnitude of the maximal tangential stress in a De Witt medium and agrees with the evolutionarity condition for the hydrodynamics equations of this system [5].

Thus, the evolutionarity condition for a model with a Jaumann derivative is derived from the Clausius inequality under the assumption of applicability of the inequality for virtual strains with arbitrary instantaneous strain structure. In particular, the local thermodynamic state should be stable upon rotation of the principal strain axes in the material.

Therefore, nonevolutionarity of the equations for the viscoelastic models considered turns out to be a consequence of incorrect assignment of the internal energy for adiabatic processes, or of the free energy for isothermal processes. The nature of the nonevolutionarity detected in the dynamics equations of a nonlinearly elastic medium is analogous [16].

However, there exists another aspect of nonevolutionarity associated with the arbitrariness in the simulation of the dissipative processes, for example, in the model of a non-Newtonian fluid with nonlinear viscosity [17]. For such media the evolutionarity conditions can evidently be consequences of the extremal principles of the thermodynamics of irreversible processes.

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